This week:
• completeness of representation in time and frequency domain
• some key properties of Fourier transforms
• filtering: convolution and deconvolution
• practical limitations to deconvolution with applications to microscopy
Representing functions: discretization and the “time basis”
Basis functions as coordinate axes

\[ f(t) = \sum_{i=1}^{N} f_i \delta(t - t_i) \]

\[ f(t) = \sum_{i=1}^{N} a_i g_i(t) \]

\[ \{ \delta(t - t_i) \} \leftrightarrow \{ g_i(t) \} \]
SIGNALS IN TIME AND FREQUENCY DOMAIN

\[ f(t) = \sum_{n} f_n \delta(t - t_n) = \sum_{n} F_n \sin(\omega_n t) \]

what needs to be true to guarantee we can transform between two basis sets in both directions?
Sinusoids as basis functions

\[ f(t) = A \cos(\omega t + \phi) = B \cos(\omega t) + C \sin(\omega t) \]

Each frequency needs 2 pieces of information, or 2 dimensions
**KEY**: The sine/cosine basis is complete and orthogonal

Completeness:

All well behaved functions can be written as a sum of sines and cosines

Orthogonality:

\[
\int dt \cos(kt) \cos(lt) = 0, \quad k \neq l
\]

\[
\int dt \sin(kt) \sin(lt) = 0, \quad k \neq l
\]

\[
\int dt \cos(kt) \sin(lt) = 0
\]
Complex numbers, sines and cosines

\[ e^{i\phi} = \cos(\phi) + i \sin(\phi) \]

\[ \text{Im}(e^{i\phi}) = \sin(\phi) \]

\[ \text{Re}(e^{i\phi}) = \cos(\phi) \]
**KEY**: The complex exponential basis is complete and orthogonal

Completeness:

All well behaved functions can be written as a sum of complex exponentials

Orthogonality:

\[
\int dt \ e^{ikt} e^{ilt} = \int dt \ e^{i(k-l)t} = \delta(k-l)
\]
The Fourier transform:

\[ F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt \]

The inverse Fourier transform:

\[ f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} \, d\omega \]
The Fourier transform returns a complex function

\[ F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt \]

We can split that into its two components:

\[ Re(F(\omega)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cos(\omega t) \, dt. \]

\[ Im(F(\omega)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \sin(\omega t) \, dt. \]

Both components are needed to capture the amplitude and phase at every frequency.

Integration against a basis function is the equivalent of a projection.
SOME EXAMPLES

\[ f(t) = \cos(\omega t) \]

\[ \text{Real}(F(\omega)) \]

\[ \text{Imag}(F(\omega)) \]

\[ \cos(\omega t) \]

\[ \sin(\omega t) \]
GAUSSIAN: width in time and frequency domain

\[ f(t) = \exp(t^2 / 2\sigma^2) \]
AMPLITUDES AND PHASES

original  phase noise  amplitude noise
PHASES AND EDGES

Note specificity of phases: all frequency components are aligned (i.e. all real and hence cosines) to create edge
Decomposition of a square wave.

Components

Cumulative sum
The power spectrum and Parseval’s theorem

\[ \sigma^2 = \frac{1}{T} \int_0^T |f(t)|^2 dt = \frac{1}{2\pi} \int_0^\infty |F(\omega)|^2 d\omega \]

- The variance in the time domain is equal to the total power in the frequency domain
- The power spectrum parses the variance into frequencies
NYQUIST FREQUENCY AND ALIASING
This is not a good thing for your data!
Filtering: Impulses responses and convolution

Dynamics instantiated by **linear** differential equations can be summarized by a filter and hence applied to an arbitrary input via convolution

\[
x(t) = \int d\tau h(\tau)y(t - \tau)
\]

\(h(\tau)\) filter or impulse response

\(x(t)\) built up from superposition of impulse responses weighted by \(y(t-\tau)\)

This could be for differential equations in time (dynamics) or space (diffusion/diffraction)

\[
f(x) = \int dx' h(x')g(x - x')
\]

how do we determine \(h\)?
General solution to second order DEQ

\[ m \frac{d^2 x(t)}{dt^2} = F(t) - k x(t) - c \frac{dx(t)}{dt} \]

\[ x(t) = \int h(\tau) F(t - \tau) d\tau \]
Application: how do we determine appropriate filter?

how do we determine $h$ for synapse between rods and rod bipolar cell?
Filtering

original

low-pass filtered

high-pass filtered