Fourier Analysis and Convolution

An Introduction

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What is a Fourier Transform?

• Representation of an arbitrary function, \( f(x) \), as a sum of pure harmonic functions

\[
f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \ldots
\]

\[
= a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)
\]

• Transformation of weights of times (or \( x \)'s) to a series of weights of frequency

\[
f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \ldots
\]

\[
f(x) = c_0 \delta(x) + c_1 \delta(x - \frac{1}{2\pi}) + c_2 \delta(x - \frac{2}{2\pi}) + \ldots
\]

Transform: \( \{c_i\} \Leftrightarrow \{a_k, b_k\} \)
Key properties of pure harmonics

- They are **orthogonal**
  \[ \int_0^{2\pi} \sin(kx) \sin(lx) \, dx = 0, \quad k \neq l \]
  \[ \int_0^{2\pi} \cos(kx) \sin(kx) \, dx = 0 \]

- They are **complete**
  No function can be orthogonal to all harmonics

- They are **convenient**
  Intuitive, simple, and calculated efficiently
Intensities at time points to “intensities” at frequencies
A signal comprised of sum of 3 sinusoids
Some important examples
Decomposition of a square wave

Components

Cumulative sum

\( k \)

3
9
15
21
27
33
39
45

\( x \) or \( t \) (0 to 2\( \pi \) or \(-\pi\) to \(\pi\))
Fourier series expansion as weights on sinusoidal basis functions

**Visualize the 0\textsuperscript{th} thru 9\textsuperscript{th} components**

\[ f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \ldots + a_9 \cos 9x + b_9 \sin 9x \]
**Fourier series expansion as weights on sinusoidal basis functions**

Visualize the $0^{th}$ thru $9^{th}$ components

\[ f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \ldots + a_9 \cos 9x + b_9 \sin 9x \]

$x$ encoding dimension: $0$ to $2\pi$
Fourier decomposition of square wave

\[
\begin{align*}
&= \sum \\
&= \sum
\end{align*}
\]
How to compute a fourier coefficient

How do you get the $a_k$ and $b_k$ in…

$$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \ldots$$

Multiply left and right sides by $\cos(kx)$ and integrate from 0 to $2\pi$

$$\int_0^{2\pi} f(x) \cos kx \, dx = \int_0^{2\pi} \cos kx \left( a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \ldots \right) \, dx$$

$$= \int_0^{2\pi} \cos kx \left( a_k \cos kx \right) \, dx$$

This simplification comes from the **orthogonal** property

$$= a_k \int_0^{2\pi} \cos^2 kx \, dx$$

$$= a_k \pi$$

Therefore

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx \, dx$$
Cosine and sine components