Differential equations and neural dynamics

First order linear differential equations
  in time
  in frequency

Second order and coupled linear DEs

Differential equations as filters

Decomposition into eigenvectors

Feedback

Nonlinear equations

Numerical methods
What is a differential equation?

Expresses the *dynamics* of a system

\[ y' = \frac{dy}{dt} = f(t, y, x, ..) \]

This week we will only talk about change in time, but DEs can also express changes in space, and both together:

\[ \frac{\partial}{\partial t} C(x, t) = D \frac{\partial^2}{\partial x^2} C(x, t) \]
Integrating a differential equation

A pure integrator:

\[
\frac{dy}{dt} = f(t)
\]

\[
f(t) = \begin{cases} 
0, & 0 < t < 10 \\
2, & 10 \leq t < 20 \\
0, & t > 20.
\end{cases}
\]

\[
y(t) = y(0) + \int_0^t dt \ f(t)
\]

\[
y(t) = \begin{cases} 
y_0, & 0 < t < 10 \\
y_0 + 2(t - 10), & 10 \leq t < 20 \\
y_0 + 20, & t > 20.
\end{cases}
\]
First order differential equations

Most important general class of equations; describe many physical processes

The rate of change of a quantity is proportional to the quantity

\[ y' = \frac{dy}{dt} = \alpha y \]

Examples..
The parallel RC circuit

Ohm’s law: \( V = I_R R \)

Capacitor: \( C = \frac{Q}{V} \)

Kirchhoff: \( I_R + I_C = I_{ext} \)

\[
C \frac{dV}{dt} = -\frac{V}{R} - I_{ext}
\]

\[
\tau \frac{dV}{dt} = -V + V_\infty
\]
Linear first order solution

... solve for $I_{\text{ext}} = 0$:

$$V(t) = V(0)e^{-t/RC}$$

$V(0)$ is the initial condition

$$\tau \frac{dV}{dt} = -V + V_\infty$$

$$V(t) = V_\infty + (V(0) - V_\infty)e^{-t/\tau}$$

Stability...
\[ V(t) = V_\infty + (V(0) - V_\infty)e^{-t/\tau} \]
With a time-varying driving current:

\[ C \frac{dV}{dt} = -\frac{V}{R} - I_{\text{ext}} \]

\[ V(t) = V(0)e^{-t/\tau} + \frac{1}{C} \int_{t'}^{t} dt' \ e^{-\left(t-t'\right)/\tau} I(t') \]
Using the Fourier transform

\[ f(t) = \frac{1}{\sqrt{2\pi}} \int d\omega \ e^{i\omega t} \tilde{f}(\omega) \]

\[ \frac{d}{dt} f(t) \rightarrow i\omega \tilde{f}(\omega) \]

\[ \frac{d^2}{dt^2} f(t) \rightarrow -\omega^2 \tilde{f}(\omega) \]
The RC circuit in the frequency domain

\[
\tau \frac{dV}{dt} = -V + RI
\]

\[
i\omega \tau \tilde{V}(\omega) = -\tilde{V}(\omega) + R\tilde{I}(\omega)
\]

\[
\tilde{V}(\omega) [i\omega \tau + 1] = R\tilde{I}(\omega)
\]

\[
\tilde{V}(\omega) = \frac{1}{1 + i\omega \tau} R\tilde{I}(\omega)
\]

\[
= \frac{(1 - i\omega \tau)}{1 + (\omega \tau)^2} R\tilde{I}(\omega)
\]
Higher order linear differential equations

A very important case: second order systems

e.g. mechanical oscillations

\[ F = ma = m \frac{d^2x}{dt^2} \]

\[ m \frac{d^2x}{dt^2} = -kx - c \frac{dx}{dt} \]

spring      viscous damping
\[ m \frac{d^2 x}{dt^2} + c \frac{dx}{dt} + kx = 0 \]

\[ x(t) = Ae^{\lambda t} \]

\[ mA\lambda^2 e^{\lambda t} + cA\lambda e^{\lambda t} + kAe^{\lambda t} = 0 \]

\[ m\lambda^2 + c\lambda + k = 0 \]

\[ \lambda = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} \]
Two solutions.. two boundary conditions

\[ x(t) = Ae^{\lambda t} \]
\[ \lambda = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} \]

- \( c^2 - 4mk > 0 \) \rightarrow Two roots with negative real part; \textit{overdamped}
- \( c^2 - 4mk = 0 \) \rightarrow Degenerate roots; \textit{critically damped}
- \( c^2 - 4mk < 0 \) \rightarrow Two complex roots; \textit{underdamped}
The overdamped case

\[ x(t) = Ae^{\lambda t} \]
\[ \lambda = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} \]
\[ c^2 - 4mk < 0 \]
Fourier solution of second order problem with forcing

\[ m \frac{d^2 x}{dt^2} + c \frac{dx}{dt} + kx = F(t) \]

\[ -m\omega^2 \ddot{x} + i\omega c \ddot{x} + k \ddot{x} = \tilde{F}(\omega) \]

\[ \ddot{x} = \frac{F(\omega)}{k - m\omega^2 + ic\omega} \]

\[ \ddot{x} = F(\omega) \frac{k - m\omega^2 - ic\omega}{[k - m\omega^2]^2 + c^2 \omega^2} \]
Fourier solution of second order problem

\[ \tilde{x} = F(\omega) \frac{k - m\omega^2 - ic\omega}{[k - m\omega^2]^2 + c^2\omega^2} \]

If constants permit, can have a peak: a certain frequency is enhanced.\[ \rightarrow \] resonance

This case is analogous to RLC circuits.
Systems of differential equations

What if we have many variables changing at once?

An important example is higher order equations!

\[ m \frac{d^2 x}{dt^2} + c \frac{dx}{dt} + kx = 0 \]

\[ y = x' \]

\[ my' + cy + kx = 0 \]

\[ \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \]
Matrix solutions

What do we do now?
Generalization of the first order equation to matrices

\[ x' = Ax \]

\[ x(t) = \sum a_i(t) \mathbf{e}_i \]

\[ \rightarrow x(t) = \sum a_i \mathbf{e}_i e^{\lambda t} \]

Solutions of the system are combinations of eigen vectors \( \mathbf{e}_i \)

Dynamics evolve along eigenvectors

Time constants determined by eigenvale \( \lambda \)

Combinations \( a_i = a_i(t = 0) \) determined by initial conditions
Matrix solution of second order equation

\[
\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\]

The dynamics are determined by the eigenvalues and eigenvectors of the system.

\[
\det \begin{pmatrix} 0 - \lambda & 1 \\ -\frac{k}{m} & -\frac{c}{m} - \lambda \end{pmatrix} = 0
\]
Solving for the eigenvalues

\[\det \left( \begin{array}{cc} 0 - \lambda & 1 \\ -\frac{k}{m} & -\frac{c}{m} - \lambda \end{array} \right) \begin{pmatrix} x \\ y \end{pmatrix} = 0\]

We obtain a familiar equation:

\[(0 - \lambda) \left( -\frac{c}{m} - \lambda \right) + \frac{k}{m} = 0\]

\[\lambda^2 + \lambda \frac{c}{m} + \frac{k}{m} = 0\]

\[m\lambda^2 + c\lambda + k = 0\]

\[\lambda = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}\]

All solutions are decaying/increasing exponentials and/or oscillations
Back to a previous example: masses and springs

\[ m \frac{d^2 x(t)}{dt^2} = -k x(t) \quad \rightarrow \quad x(t) \sim \sin(\omega t) \]

\[ \omega = \sqrt{\frac{k}{m}} \]

\[ m \frac{d^2 x(t)}{dt^2} = -K x(t) \quad \rightarrow \quad -m \omega^2 \tilde{x}(\omega) = -K \tilde{x}(\omega) \]

\[ x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \quad K = \begin{bmatrix} -2k & k \\ k & -2k \end{bmatrix} \]
Masses and springs

\[
K = \begin{bmatrix}
-2k & k \\
k & -2k
\end{bmatrix}
\]

- Eigenvectors and eigenvalues of \( K \) determine dynamics

\[
e_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad e_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]

\[
\lambda_1 = -1 \rightarrow \omega_1 = \sqrt{k/m} \quad \lambda_2 = -3 \rightarrow \omega_2 = \sqrt{3k/m}
\]
Mode decomposition of dynamics

All solutions are of the form

\[ \mathbf{x}(t) = \sum a_i(t) \mathbf{e}_i \]

\[ \mathbf{x}(t) = \sum a_i \mathbf{e}_i e^{\lambda t} \]
Networks and feedback

$$\tau \frac{d}{dt} r_1 = -r_1 + I$$

$$\tau \frac{d}{dt} r_1 = -r_1 + \lambda r_1 + I$$

$\tau_{\text{eff}} \rightarrow \frac{\tau}{1 - \lambda}$

$\tau_{\text{eff}} \rightarrow \frac{I}{1 - \lambda}$

$\lambda < 0$  
Negative feedback

Reduces time constant

Increases bandwidth

Reduces gain
Networks and feedback

\[ \tau \frac{d}{dt} r_1 = -r_1 + I \]

\[ \tau \frac{d}{dt} r_1 = -r_1 + \lambda r_1 + I \]

\[ \tau_{\text{eff}} \rightarrow \frac{\tau}{1 - \lambda} \]

\[ I_{\text{eff}} \rightarrow \frac{I}{1 - \lambda} \]

- \( \lambda > 0 \) \quad \text{Positive feedback}
- \( 0 < \lambda < 1 \) \quad \text{Stable: increased } \tau
- \( \lambda > 1 \) \quad \text{Unstable}
- \( \lambda = 0 \) \quad \text{Marginal case}
Networks and feedback

\[
\tau \frac{d}{dt} r_1 = -(1 - w_{11}) r_1 - w_{22} r_2
\]

\[
\tau \frac{d}{dt} r_2 = w_{21} r_1 - (1 - w_{22}) r_2
\]
Nonlinear feedback

\[
\frac{dx(t)}{dt} = \frac{\alpha y(t)}{(1 + \gamma x(t))^n} - \beta x(t)
\]
The Hodgkin-Huxley equations

\[ I = C_m \frac{dV}{dt} + g_K n^4 (V - V_K) + g_{Na} m^3 h (V - V_{Na}) + g_{\text{leak}} (V - V_{\text{leak}}) \]

\[ \frac{dm}{dt} = \alpha_m (1 - m) - \beta_m m \]

Some analytical solutions in special cases;

Linearization

Numerical solutions
Numerical solutions

Based on the Taylor expansion:

\[ f(x_0 + \delta x) \sim f(x_0) + \delta x f'(x)|_{x_0} \]

\[ f(x_0 + \delta x) = f(x_0) + \delta x f'(x)|_{x_0} + (\delta x)^2 f''(x)|_{x_0} + \cdots \]

Euler method
Linearization: types of solutions

two \(-\)ve real eigenvalues \rightarrow \text{stable fixed point}

two \(+\)ve real eigenvalues \rightarrow \text{unstable fixed point}

one \(+\)ve, one \(-\)ve real \rightarrow \text{saddle point}

complex eigenvalues, \(-\)ve real part \rightarrow \text{stable spiral}

complex eigenvalues, \(+\)ve real part \rightarrow \text{unstable spiral}