Fourier transforms, filtering and convolution

Adrienne Fairhall

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Quick review: linear algebra

*Linearity* – two criteria

*Linear transformation* – examples?

*Projection* – how do you do it?

*Basis set* – what is it? How big is it?

*Linear operators* – examples?

Linear transformation .. linear operators .. filters
Basis sets

• Examples?
Representing functions: discretization and the “time basis”
Basis functions as coordinate axes

\[ f(t) = \sum_{i=1}^{N} f_i \delta(t - t_i) \]

\[ f(t) = \sum_{i=1}^{N} a_i g_i(t) \]

\{ \delta(t - t_i) \} \leftrightarrow \{ g_i(t) \}
A function as a point in the time basis

The Fourier transform is a linear coordinate transformation in function space.
Sinusoids as basis functions

\[
\cos(\omega t + \phi) = \cos(\omega t) \cos(\phi) - \sin(\omega t) \sin(\phi) = A \cos(\omega t) + B \sin(\omega t)
\]

Each frequency needs 2 pieces of information, or 2 dimensions
A function as a point in the time basis

For each problem, look for a natural coordinate frame that reduces complexity.

data compression

“Sparse” basis
The Fourier transform reexpresses a function of time as a function of frequency by decomposing it into sinusoids at different frequencies.

Any (finite, square-integrable) function can be written in this way.
Decomposition of a square wave.

**Components**

**Cumulative sum**
The Fourier transform:

\[ F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt \]

The inverse Fourier transform:

\[ f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} \, d\omega \]

Integration against a basis function is the equivalent of a projection
Quick review of complex numbers

\[ e^{i\phi} = \cos(\phi) + i\sin(\phi) \]

Polar representation

Cartesian representation

\[ \text{Re}(e^{i\phi}) = \cos(\phi) \]
\[ \text{Im}(e^{i\phi}) = \sin(\phi) \]
The Fourier transform returns a **complex** function

\[
F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt
\]

\[
F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) [\sin(\omega t) + i \cos(\omega t)] \, dt.
\]

We can split that into its two components:

\[
Re(F(\omega)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cos(\omega t) \, dt.
\]

\[
Im(F(\omega)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \sin(\omega t) \, dt.
\]

Both components are needed to capture both the **amplitude** and **phase** at every frequency.
Just as the function $f(t)$ consists of a set of *coefficients* of delta functions in time,

the transform $F(\omega)$ consists of a set of *coefficients* of cosines (real part) and sines (imaginary part) of different frequencies.
At each frequency $\omega$, we have one complex number,

$$F(\omega) = \text{Re}(F(\omega)) + i \text{Im}(F(\omega)) = F_\omega e^{i\phi}$$

The **amplitude** at that frequency is then $|F(\omega)|$;

The **phase** is $\phi = \tan^{-1}\left(\frac{\text{Im}(F(\omega))}{\text{Re}(F(\omega))}\right)$

The **power** at $\omega$ is the square amplitude $|F(\omega)|^2$
The power spectrum and Parseval’s theorem

\[
\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega
\]

The variance in the time domain is equal to the total power in the frequency domain.
The sine/cosine basis is complete and orthogonal

Completeness:

All well behaved functions can be written as a sum of sines and cosines

Orthogonality:

\[ \int dt \, e^{ikt} e^{ilt} = \int dt \, e^{i(k-l)t} = \delta(k-l) \]

\[ \int dt \, \cos(kt) \cos(lt) = 0, \quad k \neq l \]

\[ \int dt \, \sin(kt) \sin(lt) = 0, \quad k \neq l \]

\[ \int dt \, \cos(kt) \sin(lt) = 0 \]
Decomposition of a square wave.
First example: FT of $\cos(\omega t)$
FT of cosine and sine
More examples
The frequency range of the Fourier transform

• How much information is there? How many frequencies are there?

• What is the lowest frequency you can detect in your data?

• What is the highest frequency?
The lowest represented frequencies: wrapping

The lowest nonzero frequency one can obtain from the FFT is $1/T$. 
The Nyquist frequency

There is only one value (real) at the Nyquist frequency
The Nyquist frequency
The Nyquist frequency
The Nyquist frequency
Nyquist and aliasing
Filtering, convolution and transfer functions

- Filtering in the time domain
- Transforming to the frequency domain: the transfer function
- Equivalence between convolution and multiplication
- Some examples
- Combining filters
- Impulse response function
- White noise analysis
Linear systems

What is a linear system?

The output depends *linearly* on the input.

The output is a *linear combination* of the input, at different times or, equivalently, at different frequencies.

.. systems described by a linear differential equation
Linear filtering in the time domain

\[ g(t) = \sum_j h_j f(t_{i-j}) \]

The set of coefficients \( h_j \rightarrow h(t) \) is called a filter (or a kernel, or a Greens function, or a point spread function).
\[ g(t) = h \ast f = \int d\tau \ h(\tau) \ f(t - \tau) \]
Filtering in the frequency domain

The convolution theorem

\[ g(t) = \int d\tau \ h(\tau) f(t - \tau) d\tau \]
\[ = \int d\tau \ \left( \int d\omega \ H(\omega) e^{i\omega \tau} \right) \left( \int d\omega' \ F(\omega') e^{i\omega'(t-\tau)} \right) \]
\[ = \int d\tau \int d\omega \int d\omega' \ e^{i\omega \tau} e^{i\omega'(t-\tau)} H(\omega) F(\omega') \]
\[ = \int d\omega \int d\omega' \left[ \int d\tau e^{i(\omega - \omega')\tau} \right] e^{i\omega' t} H(\omega) F(\omega') \]
\[ = \int d\omega \int d\omega' \ \delta(\omega - \omega') e^{i\omega' t} H(\omega) F(\omega') \]
\[ = \int d\omega \ e^{i\omega t} H(\omega) F(\omega) \]

\[ G(\omega) = H(\omega) F(\omega) \]
Filtering in the frequency domain

output \[ G(\omega) = H(\omega) F(\omega) \] input

Transfer function: \[ H(\omega) = G(\omega) / F(\omega) \]

A filter can throw away information; it can’t introduce new information.
Color vision as filtering in the frequency domain
Filtering in space

Example: point spread function

\[ I(x_i, y_i) = \int \int O(u, v) \text{ PSF}(x_i - Mu, y_i - Mv) \, du \, dv \]
Image processing

Lowpass

Highpass