The Fourier transform:

\[ F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt \]

The inverse Fourier transform:

\[ f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} \, d\omega \]
The Fourier transform returns a complex function of frequency

\[ F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)[\cos(\omega t) + i \sin(\omega t)] dt \]

We can split that into its two components:

\[ Re(F(\omega)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cos(\omega t) dt. \]

\[ Im(F(\omega)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \sin(\omega t) dt. \]

Both components are needed to capture both the amplitude and phase at every frequency.
How much information?

Number of frequencies: $f_{\text{min}}$ to $f_{\text{max}}$

How many frequency coefficients?

How many numbers altogether?

How to reconcile?

Even..
Odd..
Fourier transforms and linear operators

\[ f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega \]

What is the derivative of \( f(t) \)?

...... second derivative?

...... integral?
Let’s solve a differential equation both ways.

\[ C \frac{dV_m}{dt} = I_{inj} + \frac{(V_m - V_r)}{R} \]
Now in the frequency domain

\[ i\omega \tilde{V}(\omega) = \frac{1}{\tau}(-\tilde{V}(\omega) + R\tilde{I}(\omega)) \]

\[ \tilde{V}(\omega) = \frac{R}{1 + i\omega \tau} \tilde{I}(\omega) \]

\[ = \frac{R(1 - i\omega \tau)}{1 + (\omega \tau)^2} \tilde{I}(\omega) \]
Second order systems

\[ m \frac{d^2 x}{dt^2} + c \frac{dx}{dt} + kx = F(t) \]

\[-m\omega^2 \ddot{x} + i\omega c \dot{x} + k \ddot{x} = \tilde{F}(\omega) \]

\[ \ddot{x} = F(\omega) \frac{k - m\omega^2 - ic\omega}{[k - m\omega^2]^2 + c^2\omega^2} \]
Eigenvectors and eigenvalues

A way to reorganize your coordinate space so that

\[ M \mathbf{x} = a \mathbf{x} \]
Fourier modes are the **eigenvectors** of time-translationally invariant systems

$$\frac{d}{dt} \rightarrow \text{Fourier basis diagonalizes linear differential equations}$$
Fourier modes are the **eigenvectors** of time-translationally invariant systems

\[ h \ast f = \int d\tau \ h(\tau) \ f(t - \tau) \]

\[ \begin{bmatrix}
    h_1 & 0 & 0 & 0 & 0 & 0 \\
    h_2 & h_1 & 0 & 0 & 0 & 0 \\
    0 & h_2 & h_1 & 0 & 0 & 0 \\
    0 & 0 & h_2 & h_1 & 0 & 0 \\
    0 & 0 & 0 & h_2 & h_1 & 0 \\
    0 & 0 & 0 & 0 & h_2 & h_1 \\
\end{bmatrix} \]

\[ \rightarrow \text{Fourier basis diagonalizes } \text{filtering} \text{ operations} \]

\[ G(w) = H(w) \ F(w) \]
• All linear differential equations are expressible as filters
• Not all filters are equivalent to differential equations!
Filtering in the frequency domain

output \[ G(\omega) = H(\omega) F(\omega) \] input

Transfer function: \[ H(\omega) = G(\omega) / F(\omega) \]

A filter can throw away information; it can’t introduce new information.
Color vision as filtering in the frequency domain

Massive dimensionality reduction!
Filtering in space

Example: point spread function

\[ I(x_i, y_i) = \int \int O(u, v) \text{ PSF}(x_i - Mu, y_i - Mv) \, du \, dv \]
Image processing

Lowpass  Highpass
Combining filters

This is easiest to see in the frequency domain.

\[ G(\omega) = H(\omega) I(\omega) \]

Let’s now make an additional linear operation on the output:

\[ G'(\omega) = H'(\omega) G(\omega) \]

Then

\[ G'(\omega) = H'(\omega) H(\omega) I(\omega) = K(\omega) I(\omega) \]

In the time domain, the new filter is the convolution:

\[ k(t) = h' * h \]

*Note: order doesn’t matter.*
How do you figure out what your filter is?  
(Linear systems analysis)

Time domain: the impulse response function

\[ g(t) = h * f = \int d\tau \ h(\tau) f(t - \tau) \]

What if we put in \( f(t) = \delta(t - \tau) \)?

\[ \rightarrow \] the output is \( h(t) \).
How do you figure out what your filter is?

Frequency domain: white noise analysis

Transfer function

\[ H(\omega) = \frac{G(\omega)}{F(\omega)} \]

A nice input to use is then one with a flat spectrum, \( F(\omega) = \text{const} \).

What happens if the noise is not white (or the impulse is not a perfect delta function)?
Getting spectra you can trust
Finite data size effects

Data = f(t) \times W(t)
The sinc function

\[ f(x) = \frac{\sin(\pi x)}{\pi x} \]

Width \(\sim 1/T\)
Example: a pure frequency

\[ F(\omega) \]
Problem #2: averaging
Problem #2: averaging

\[ \delta \omega \rightarrow \delta \omega / N \]
Optimal windowing
Optimal windowing

Tradeoff between independent samples and frequency resolution

\[ p = 2 T \Delta \omega - 1 \]
Tapers

Pesaran
A. Single trial, 500ms periodogram

B. Single trial, 10Hz multitaper estimate

C. Single trial, 20 Hz multitaper estimate

D. Nine-trial average, 10 Hz multitaper

Pesaran
Windowed spectrum of a pure sine wave

Diagram A:
- Time series
- Taper
- Product

Diagram B:
- Leakage
- Central band
- Idealized
- Half-sine taper
- No taper
- Single Slepian taper

Normalized [Fourier transform] vs. Frequency
Stationarity: time-frequency analysis
Wavelets and compression

\[ \Phi_{a,b}(x) = \frac{1}{\sqrt{a}} \Phi \left( \frac{x - b}{a} \right) \]
Sparseness
Lossy compression
JPEG

Decreasing number of coefficients $\rightarrow$ Decreasing quality
Visual receptive fields: what is optimized?

Olshausen and Field, Bell and Sejnowski, Lee
Correlation and coherence

Autocorrelation  $\rightarrow$  Spectral density
Cross-correlation  $\rightarrow$  Cross-spectral density
Coherence

\[ C(\omega) = \frac{R(\omega)S^*(\omega)}{|R(\omega)||S(\omega)|} \]
Example: leech circuitry
Who is being driven by the input?

Taylor, Cottrell, Kleinfeld and Kristan, 2003