A differential equation describes the change in neural responses (or ionic concentrations, etc.) between the present time \( t \) and a time \( (t + \text{d}t) \), which lies infinitesimally in the future, and it describes this change as some function of all the physiologically relevant variables at time \( t \). The order of a differential equation is defined as the highest derivative present in the equation. As will be seen in the next chapter, this is equivalent to the number of coupled first order equations in a system of differential equations. In this chapter I introduce the simplest differential equation of major importance in science: the first order linear differential equation with constant coefficients. Solutions of this equation form the basis for understanding higher order equations, both linear and non-linear. Despite its linearity, however, the first order equation is still capable of describing the spike rate of a single neuron in response to stimulation, and we shall see that it can also describe interactions between postsynaptic potentials.

2.1 The fundamental first order equation

Let us begin our treatment of differential equations by considering the simplest, yet most fundamental of all differential equations in the sciences:

\[
\frac{\text{d}x}{\text{d}t} = -\frac{1}{\tau}x
\]

(2.1)

This equation states that the rate of change of the function \( x(t) \) as a function of time \( t \) is equal to a constant times the function itself. The time constant, \( \tau \), can be shifted to the left side of the equation by simple multiplication, but the form (2.1) will be useful for our present purposes. The temporal units of \( \tau \) are the same as \( t \) (i.e. milliseconds, ms, or fraction of a second). We can solve (2.1) by substituting an exponential function and then making use of its derivative from (1.2):

\[
x(t) = A e^{at}, \text{ so } \frac{\text{d}x}{\text{d}t} = A a e^{at}
\]

Substitution of this into (2.1) leads to the result:

\[
A a e^{at} = -\frac{A}{\tau} e^{at}
\]
This equation is easily solved to give $a = -1/\tau$, so the solution to (2.1) is just $x = A \exp(-t/\tau)$. The minus sign on the right side of (2.1) results in a negative value for the exponent in the solution ($\tau$ is always positive by convention). As a negative exponent is generally the most appropriate physiological solution, it has been emphasized, although the same derivation obviously holds for positive values on the right-hand side. The constant $A$ may be assigned any value we wish. To determine a unique value of $A$, we need to specify an initial condition, namely, a value for $x(t)$ at $t = 0$. Denoting this value by $x_0$ and substituting into the solution at $t = 0$, we see that the solution to (2.1) is:

$$x(t) = x_0 e^{-t/\tau}$$  \hspace{1cm} (2.2)

Let us now consider a somewhat more complex equation in which the right-hand side contains an arbitrary additive function $S(t)$. As we shall see shortly, this may be thought of as a time-varying stimulus applied to a neuron which responds with spike rate $x(t)$. The relevant equation is:

$$\frac{dx}{dt} = \frac{1}{\tau}(-x + S(t))$$  \hspace{1cm} (2.3)

Equation (2.1) is referred to as a homogeneous differential equation, because the right side only contains terms involving the unknown function $x$. Equation (2.3), on the other hand, is termed inhomogeneous, because the right-hand side contains an additional term that is independent of $x$. To solve (2.3), let us try to find a solution of the form:

$$x = A e^{-t/\tau} + H(t) e^{-t/\tau}$$  \hspace{1cm} (2.4)

This is just the solution we obtained to (2.1) plus an additional function of time, $H(t)$, multiplied by the exponential that was obtained from solving (2.1).

Substitution of (2.4) into (2.3) yields:

$$e^{-t/\tau} \frac{dH}{dt} - \frac{1}{\tau} (A e^{-t/\tau} + H e^{-t/\tau}) = \frac{1}{\tau} (S - A e^{-t/\tau} - H e^{-t/\tau})$$

Cancellation of the second and third terms on both sides gives:

$$e^{-t/\tau} \frac{dH}{dt} = \frac{1}{\tau} S$$

This may be solved for $H(t)$ by integration with the result:

$$H(t) = \frac{1}{\tau} \int_0^t e^{t'/\tau} S(t') \, dt'$$  \hspace{1cm} (2.5)

The last step results from isolating $dH/dt$ on the left and then integrating both sides of the equation. This integration is carried out with respect to the 'dummy variable' $t'$, which
represents all past times between the start of stimulation at \( t = 0 \) and the present time \( t \). Combining (2.2), (2.4), and (2.5), we have proved the theorem:

**Theorem 1:** The solution to the equation:

\[
\frac{dx}{dt} = \frac{1}{\tau} (-x + S(t))
\]

is

\[
x(t) = A e^{-t/\tau} + \frac{1}{\tau} e^{-t/\tau} \int_0^t e^{(t-t')/\tau} S(t') \, dt'
\]

or

\[
x(t) = A e^{-t/\tau} + \frac{1}{\tau} \int_0^t e^{-(t-t')/\tau} S(t') \, dt'
\]

where \( A \) is chosen to satisfy the initial condition.

To obtain the second form for \( x(t) \) in this theorem, the exponential has simply been moved inside the integral. Many readers will recognize the integral in the solution as a temporal convolution integral. This convolution adds up effects throughout the past history of stimulation, \( S(t') \), weighting each past instant \( t' \) by an exponentially decaying function of the elapsed time between \( t' \) and the present, \( \exp\{- (t-t')/\tau\} \). Thus, the influence of all previous stimulation is summed, but it dies out exponentially as we move further into the past.

To fix ideas, let us apply Theorem 1 to an example. For time in milliseconds, solve the equation:

\[
\frac{dx}{dt} = \frac{1}{10} (-x + 50)
\]

for an initial condition \( x(0) = 0 \). Using Theorem 1 we obtain:

\[
x(t) = A e^{-t/10} + \frac{1}{10} \int_0^t e^{-(t-t')/10} 50 \, dt'
\]

The integral is easily evaluated with the result:

\[
\frac{1}{10} \int_0^t e^{-(t-t')/10} 50 \, dt' = 50 \left(1 - e^{-t/10}\right)
\]
Fig. 2.1 Solutions of equation in text for two different initial conditions.

Therefore:

$$x(t) = A e^{-t/10} + 50(1 - e^{-t/10})$$

At $t = 0$, $x = A$, so the initial condition will be satisfied if $A = 0$. This produces the solution:

$$x(t) = 50(1 - e^{-t/10})$$

As a second case, let us solve the same equation again but with a different initial condition: $x(0) = 70$. Now $A = 70$ and

$$x(t) = 70 e^{-t/10} + 50(1 - e^{-t/10}) = 50 + 20 e^{-t/10}$$

Both of these solutions are plotted in Fig. 2.1, where it is easy to see that $x(t)$ approaches the value 50 with a decaying exponential time course. The time constant for this approach is 10 ms, and it is apparent that the solution has virtually reached its asymptotic value by $t = 40$ ms, i.e. within about 4 time constants. This is because $e^{-4} = 0.018$, within 2% of the asymptotic value.

To take another example, let us solve the following equation for $x(0) = 0$:

$$\frac{dx}{dt} = \frac{1}{20} (-x + 40 e^{-t/20})$$

Using Theorem 1, we obtain:

$$x(t) = A e^{-t/20} + \frac{40}{20} e^{-t/20} \int_0^t e^{t'/20} e^{-t'/20} dt' = A e^{-t/20} + 2t e^{-t/20}$$
As \( x(0) = 0 \), the solution is:

\[
x(t) = 2t e^{-t/20}
\]

This function is graphed in Fig. 2.2. Note that the response \( x(t) \) overshoots the stimulus \( S(t) \) and then follows its exponential decay.

2.2 Cascades of first order equations

There are many physiological circumstances in which one first order differential equation provides the input to a second, the second to a third, and so forth. Frequently it is assumed on the basis of experimental evidence that each equation has the same time constant \( \tau \) as the others. This is known as a cascade of equations, and it generally arises when there is a chain of chemical steps between an initial event and a final measured neural response. For example, the electrical response of photoreceptors known as rods in the primate retina are well described by a three-equation cascade. Let us derive the solution to such a cascade for the rod with stages \( x, y, \) and \( z \) obeying the equations:

\[
\begin{align*}
\frac{dx}{dt} & = -\frac{1}{\tau} x \\
\frac{dy}{dt} & = \frac{1}{\tau} (-y + kx) \\
\frac{dz}{dt} & = \frac{1}{\tau} (-z + ky)
\end{align*}
\]

We will assume that several light quanta have just been captured so that \( x(0) = 1 \), but \( y(0) = 0, z(0) = 0 \). The constant \( k \) describes amplification by biochemical events in the rods. This is actually a third order differential equation (see Chapter 3), but it is so simple

\[
\begin{align*}
\frac{dx}{dt} & = -\frac{1}{\tau} x \\
\frac{dy}{dt} & = \frac{1}{\tau} (-y + kx) \\
\frac{dz}{dt} & = \frac{1}{\tau} (-z + ky)
\end{align*}
\]
that we can solve it exactly using Theorem 1. We already know the solution to the first equation, which we can then substitute into the second equation:

\[ x(t) = e^{-t/\tau} \quad \text{so} \]
\[ \frac{dy}{dt} = \frac{1}{\tau}(-y + k e^{-t/\tau}) \]  
(2.7)

The equation for \( y \) is now just a more general case of our last example, and we can again solve using Theorem 1 to get:

\[ y(t) = k \frac{t}{\tau} e^{-t/\tau} \]  
(2.8)

Substituting this expression for \( y \) in the final equation in (2.6) and again using Theorem 1 gives our final result:

\[ z(t) = \frac{k^2}{2} \left( \frac{t}{\tau} \right)^2 e^{-t/\tau} \]  
(2.9)

Thus our three-equation cascade in eqn (2.6) produces a response proportional to \( t^2 \) for small \( t \). In comparing data with eqn (2.9) it is usually convenient to plot the data on double logarithmic coordinates, because these coordinates transform the rising phase of the response into a straight line with a slope of 2.0. Figure 2.3 illustrates such a plot of human rod electroretinogram (ERG) data (Hood and Birch, personal communication).

![Double logarithmic plot of human rod ERG data](image)

**Fig. 2.3** Double logarithmic plot of human rod ERG data (from Hood and Birch, 1990). As shown by the solid line, the initial phase of the response has a slope of 2.0 in agreement with eqn (2.9).
As shown by the solid line, the rod ERG response has an initial slope of 2.0, so there are three stages in the rod biochemical response to light as described in (2.6) and solved in (2.9). This shows that a neuroscientist can sometimes infer aspects of the underlying biochemistry or circuitry from a theoretical analysis of the measured response.

We can generalize our treatment of cascades to \( N \) stages, each with time constant \( \tau \) and amplification \( k \) to yield:

\[
z_N(t) = \frac{k^{N-1}}{(N-1)!} \left( \frac{t}{\tau} \right)^{N-1} e^{-t/\tau}
\]  

(2.10)

where \( z_N(t) \) is the response of the \( N \)th stage of the reaction cascade. Note that we can allow different values of the amplification \( k \) so long as \( \tau \) remains constant for all stages.

### 2.3 Responses of a simple model neuron

Let us now see how (2.3) can be modified to describe the response of a simple neuron to an external stimulus. This neuron will be represented by its spike rate as a function of time without describing the shape and timing of each individual spike. Before diving into the mathematics, however, a brief discussion of neural responses as a function of stimulus intensity is in order.

The particular physiological example I shall choose is from the visual system, but similar functional relationships with slightly different parameter values are common in the nervous system. Sclar, Maunsell, and Lennie (1990) measured the spike rate of visual neurons in response to stimuli of varying contrast or intensity. Recordings from several different levels of the visual system (lateral geniculate, striate cortex, middle temporal cortex) showed that all neurons could be described by a single equation in which only the parameters differed among visual areas. Albrecht and Hamilton (1982) have also found that this same equation provided a better fit to their data than several other candidate equations. The equation is known as the Naka–Rushton (1966) function in vision research and as the Michaelis–Menten equation in chemical kinetics. This equation relates a stimulus intensity \( P \), which may be thought of as the net postsynaptic potential reaching the site of spike generation, to response or spike rate \( S(P) \) as follows:

\[
S(P) = \begin{cases} 
\frac{MP^N}{\sigma^N + P^N} & \text{for } P \geq 0 \\
0 & \text{for } P < 0 
\end{cases}
\]  

(2.11)

In this equation \( M \) is the maximum spike rate for very intense stimuli, and \( \sigma \) determines the point at which \( S(P) \) reaches half of its maximum. Hence, \( \sigma \) is termed the semi-saturation constant. Finally, \( N \) determines the maximum slope of the function, or how sharp the transition is between threshold and saturation. These points will become evident by inspection of Fig. 2.4, where \( M = 100, \sigma = 50, \) and \( N \) assumes several values within the range reported for visual neurons. In particular, Sclar et al. (1990) reported that lateral geniculate neurons were best fit by values of \( N \) averaging 1.4, visual cortical neurons had \( N \) values around 2.4, and middle temporal cortex neurons had \( N \) values around 3.0.
Fig. 2.4 Naka–Rushton function (2.11) plotted for three values of $N$ with $\sigma = 50$ and $M = 100$ (top panel). The bottom panel shows spike rates of four different neurons along with fits of (2.11) to the response rate of each (reproduced with permission, Albrecht and Hamilton, 1982).

Similarly, Albrecht and Hamilton (1982) reported average values of $N = 3.4$ and $M = 120$. Representative data from the Albrecht and Hamilton (1982) study are plotted in Fig. 2.4 for comparison. In this book, we shall usually let $M = 100$ and $N = 2$ for mathematical convenience. This means that our Naka–Rushton function will have an accelerating nonlinearity near $x = 0$ and will have a maximum response rate of 100 spike/s. However, none of our conclusions depend on these particular choices. The semi-saturation constant $\sigma$ will be varied to suit particular mathematical or physiological contexts. It is important to be aware that (2.11) represents the asymptotic or steady state firing rate of a neuron. As we shall see, neural responses will generally vary over time as they approach the rate determined by (2.11). Note that the general form of $S(P)$ involves a threshold for $P$ near zero followed by a roughly linear region in which $S(P)$ increases proportionally to $P$. Finally, the spike rate saturates for large $P$. Many mathematically similar functions have been used in describing neurons, particularly the hyperbolic tangent, tanh. However, all of these functions have the same general sigmoidal, or S-like
shape, which is exhibited by $S(P)$ in Fig. 2.4. As the Naka–Rushton function in (2.11) provides the best fit to physiological data (Albrecht and Hamilton, 1982), it will be used here.

We can now write down a differential equation describing the response of a single neuron to an arbitrary stimulus. If the Naka–Rushton function with $M = 100, N = 2$, and $\sigma = 40$ is designated by $S(P)$ and the neural response or spike rate is designated by $R$, then:

$$\frac{dR}{dt} = \frac{1}{\tau}(-R + S(P))$$  \hspace{1cm} (2.12)

Let us solve this equation as a function of $t$ on the assumption that $R(0) = 0$. From Theorem 1, the solution may be seen to be:

$$R(t) = A e^{-t/\tau} + \frac{1}{\tau} \int_{0}^{t} e^{-(t-t')/\tau} S(P(t')) \, dt'$$ \hspace{1cm} (2.13)

This equation describes the responses of cortical cells to a wide variety of time-varying stimuli. We can obtain exact results for the special case where $P$ is a constant input. Then eqn (2.13) may be solved by simple integration yielding:

$$R(t) = (1 - e^{-t/\tau})S(P)$$ \hspace{1cm} (2.14)

$R(t)$ is plotted for $P = 40, 80,$ and $120$ in Fig. 2.5, where $\tau = 15$ ms. Note that $S(P)$ is similar to the solid curve in the top panel of Fig. 2.4. Thus, we have solved the equation for time evolution of the spike rate of a neuron that has the nonlinear stimulus–response relationship given by $S(P)$ but that nonetheless is governed by linear dynamics. The nonlinearity in this simple neuron is apparent in Fig. 2.5 from the fact that stimulus magnitudes differing by 40 produce a progressive compression in the asymptotic response levels. This example raises the important point that the solution of a differential equation can be a nonlinear function of its input, as $S(P)$ is indeed a nonlinear function of $P$, even

![Fig. 2.5](image)

**Fig. 2.5** Solution (2.14) to eqn (2.12) for stimulus levels $P = 40, 80$, and 120.
though the dynamics in (2.12) are linear in the response $R(t)$. When we speak of a
differential equation as being linear, we really mean only that it is linear in the dynamical
variables.

One more important point may be made concerning eqn (2.12). In the case where $S(P(t)) = S$, a constant independent of $t$, we can solve immediately for the equilibrium
state of the neuron. At equilibrium, there is no variation with time, so $dR/dt = 0$. It
follows from (2.12) that:

$$ R = S \quad \text{when } \frac{dR}{dt} = 0 $$

(2.15)

The equilibrium state is also referred to as an equilibrium point or steady state, and these
terms will be used interchangeably hereafter. The important concept to retain here is that
an equilibrium point is defined mathematically as a state of the system where nothing
changes with time. We shall subsequently see that these points are extremely important to
our understanding of nonlinear neural systems. From the solution to (2.12) in (2.14), we
can see that $R(t)$ approaches equilibrium exponentially as $t \to \infty$. Therefore, this equi-
librium point is asymptotically stable. Equilibrium points and stability will be explored in
detail in the next chapter.

2.4 Excitatory and inhibitory postsynaptic potentials

With the background gained thus far, we can now examine the role of ion channels in the
generation of postsynaptic potentials. Ion channels are governed by Ohm’s law, which
states that $I = g(V - E)$, where $I$ is the ionic current across the nerve membrane, $g$ is the
conductance (reciprocal of the resistance) usually in units of nano-Siemens (nS), and $V$ is
the voltage difference across the membrane in millivolts (mV). The effective voltage for
any ionic current is the difference between the membrane potential $V$ and the equilibrium
potential $E$ of the ionic species in question. For any ionic species, the reversal potential
is identical to $E$, because that is the membrane potential at which the sign of the current $I$
changes. The equilibrium potential $E$ is determined by the Nernst equation, which
states that:

$$ E = \frac{RT}{zF} \ln \left( \frac{C_{out}}{C_{in}} \right) $$

(2.16)

where $z$ is the charge on the ion in question, and $C_{out}$ and $C_{in}$ are the respective concen-
trations of the ion outside and inside the cell. $R$ and $F$ are respectively the thermo-
dynamic gas constant and the Faraday constant, and $T$ is the temperature in degrees
Kelvin. At 20°C the ratio $RT/F = 25$ mV. The Nernst equation can be derived from
thermodynamic principles.

Let us consider a simple patch of dendrite that has a passive current due to ionic leakage
through the membrane plus channels for excitatory and inhibitory postsynaptic poten-
tials (EPSPs and IPSPs). Due to the capacitive properties of the lipid bilayer of the
Fig. 2.6 Postsynaptic responses to a single presynaptic spike for various levels of the postsynaptic resting potential from $+20 \text{ mV}$ to $-81 \text{ mV}$ (reproduced with permission, Huettner and Baughman, 1988). The reversal or equilibrium potential is close to $0.0 \text{ mV}$.

membrane, the equation for the change of membrane potential $V$ with time is:

$$\frac{dV}{dt} = -\frac{1}{\tau} \{g_i(V - E_i) + g_e(V - E_e) + g_l(V - E_l)\} \quad (2.17)$$

In this equation, $g_i$ and $E_i$ are the conductance and equilibrium potential for the leakage current; $g_e$ and $E_e$ refer to EPSP ion channels; and $g_i$ and $E_i$ are the conductance and equilibrium potential for the IPSP channels. The IPSPs simulated here are due to GABAa channels, which are present on all cortical neurons (Gutnick and Moody, 1995), including those of human neocortex (McCormick, 1989). The equilibrium potential for our excitatory synapse may be obtained from the data in Fig. 2.6, which shows EPSPs generated at a synapse for different values of the resting membrane potential $V$ (Huettner and Baughman, 1988). It is apparent that the reversal potential occurs close to $0 \text{ mV}$, which is the value we shall adopt. Similarly, the data in Fig. 2.7 show that the early GABAa equilibrium potential (due to $\text{Cl}^-$ ions) is about $-75 \text{ mV}$ in human neurons (McCormick, 1989). The average resting potential for human neurons is also about $-75 \text{ mV}$ (Avoli et al., 1994). If we set $g_i = 1 \text{ nS}$ and $\tau = 12.5 \text{ ms}$, (2.17) becomes:

$$\frac{dV}{dt} = -\frac{1}{12.5} \{(V + 75) + g_e V + g_l(V + 75)\} \quad (2.18)$$

When there is no transmitter release at either excitatory or inhibitory synapses, $g_e = 0$ and $g_i = 0$, and it is easy to see that $V = -75 \text{ mV}$ is the equilibrium value of (2.18). Suppose
Fig. 2.7  IPSP amplitudes in a human cortical neuron plotted as a function of postsynaptic resting potential (reproduced with permission. McCormick, 1989). The GABAa reversal potential (early IPSP) occurs at about \(-75\) mV. The late IPSP, due to a GABAb synapse, has a lower resting potential of about \(-95\) mV.

now that an excitatory synapse becomes active. We represent this by letting \(g_e = 2\) nS for a period of 1.0 ms. Equation (2.18) thus becomes:

\[
\frac{dV}{dt} = \frac{-1}{12.5} \left\{ (V + 75) + 2V \right\} = \frac{-1}{12.5} (3V + 75) \quad \text{for} \quad 0 \leq t \leq 1
\]

\[
V(0) = -75
\]

Thus,

\[
\frac{dV}{dt} = \frac{3}{12.5} (-V - 25)
\]

where it is assumed that \(V\) is at the resting potential before the EPSP. Using Theorem 1 the solution may be derived easily:

\[
V(t) = A e^{-0.24t} - \frac{3}{12.5} e^{-0.24t} \int_0^t e^{0.24t'} 25 \, dt'
\]

where \(0.24 = 3/12.5\). Evaluation of the integral gives:

\[
V(t) = -75 e^{-0.24t} - 25(1 - e^{-0.24t}) = -50 e^{-0.24t} - 25
\]

where \(A\) has been chosen so that \(V(0)\) satisfies the initial condition. After 1.0 ms, \(V(t) = -64.3\) mV, so we see that this conductance change due to synaptic transmission has depolarized the neuron by 10.7 mV, which is just about the EPSP amplitude obtained for a membrane potential of \(-81\) mV in Fig. 2.6. Equation (2.20) is plotted as a solid line
for the first millisecond in Fig. 2.8. (The line appears almost straight only because 1.0 ms is such a short time relative to the time constant of 12.5 ms.)

After 1.0 ms, we assume that the effects of the neurotransmitter are terminated, so \(g_e = 0\) again (synaptic conductance changes do not end quite so abruptly, but this is still a reasonable approximation). To solve for the decay in the postsynaptic potential, therefore, we must now solve the equation:

\[
\frac{dV}{dt} = -\frac{1}{12.5} \{V + 75\} \quad \text{for } t > 1
\]

\[
V(1) = -64.3
\]

Note that our new initial condition is \(V(1) = -64.3\) mV, which is not the equilibrium value. Note also that this initial condition occurs at \(t = 1\) ms, because that is the time at which the synaptic conductance reverts to its original value. To shift the initial condition from 0 to 1 ms, it is only necessary to replace \(t\) by \((t - 1)\) in the solution, so Theorem 1 produces the result:

\[
V(t) = -64.3 e^{-0.08(t-1)} - 75(1 - e^{-0.08(t-1)}) = 10.7 e^{-0.08(t-1)} - 75
\]

for \(t \geq 1\) ms. This function is also plotted in Fig. 2.8 beginning at \(t = 1\) ms. There is a striking similarity between our mathematical EPSP and the shape of the EPSP obtained in the \(-81\) mV resting condition shown in Fig. 2.6. Note that the decay back to equilibrium in (2.22) is three times slower than the initial rise of the potential in (2.20). This is a mathematical consequence of membrane conductance changes during synaptic transmission, and it indicates that this problem is actually nonlinear! We have solved for the EPSP in this case using what is called a piecewise linear approximation, which is the simplest approximation to an inherently nonlinear dynamical problem.

Let us see just how nonlinear synaptic interactions typically are. Suppose that the dendrite is at the \(-75\) mV resting potential, and synaptic activation causes a change from
Spikes, decisions, and actions

$g_i = 0$ to $g_i = 12$ nS for 1.0 ms. Referring back to eqn (2.18), you will see that this will produce no change in membrane potential, because $-75$ mV is both the resting potential and the equilibrium potential for the GABAa synapse. Can an inhibitory synapse that has no effect by itself reduce the excitation produced by an EPSP? To answer this question, let us assume that an EPSP and an IPSP occur simultaneously in the dendrite. Now we must solve (2.18) with $g_e = 2$ nS and $g_i = 12$ nS for $0 \leq t \leq 1$:

$$\frac{dV}{dt} = -\frac{1}{12.5} \left((V + 75) + 2V + 12(V + 75)\right) = -\frac{1}{12.5} (15V + 975)$$

(2.23)

The solution is again found using Theorem 1:

$$V(t) = -10 e^{-1.2t} - 65$$

(2.24)

Now the peak depolarization at $t = 1$ ms due to the EPSP has dropped to $V = -68$ mV. This is a drop from the peak of $-64.3$ mV when the EPSP occurred without a concurrent IPSP. Thus, an IPSP that has no effect on the membrane potential when it occurs alone has reduced the effect of a concurrent EPSP by about 35%. As a result of this effect, GABAa synapses are frequently termed shunting synapses because their effect is to shunt or short circuit the depolarizing current produced by EPSPs. This highly nonlinear interaction is essentially divisive rather than subtractive, although neural modelers sometimes assume that inhibition is inherently subtractive. Only when synapses are fairly far apart on the dendritic tree or on different dendritic branches do EPSPs and IPSPs interact in a manner approximating addition and subtraction.

2.5 Exercises

1. Solve the following equation for the initial condition $x(0) = 17$.

$$\frac{dx}{dt} = \frac{1}{13} (-x + 5)$$

2. Solve the following equation for the response rate $R(t)$ of a neuron for each of the following values of the postsynaptic potential: $P = 10, 20, \text{ and } 30$. Plot your results on a single graph for times up to 100 ms assuming that $R(0) = 5$ in each case.

$$\frac{dR}{dt} = \frac{1}{20} \left( -R + \frac{50P^4}{15^4 + P^4} \right)$$

3. Prove that eqn (2.9) follows from (2.6) and (2.8) by solving using Theorem 1. Now add a fourth stage, call it $w(t)$ to (2.7) governed by the equation:

$$\frac{dW}{dt} = \frac{1}{\tau} (-W + kz)$$

Derive the solution $w(t)$ for $w(0) = 0$. Does this agree with eqn (2.10)?
4. In this problem we will explore a piecewise linear approximation to action potential generation. There are two ionic channels, one for Na\(^+\) \((E_{Na} = 50 \text{ mV})\) and one for K\(^+\) \((E_K = -90 \text{ mV})\), resulting in the equation:

\[
\frac{dV}{dt} = -4\{g_{Na}(V - 50) + g_K(V + 90)\}
\]

where \(g_K = 0.6\) is the K\(^+\) resting conductance and \(g_{Na} = 0.1\) is the Na\(^+\) resting conductance. First solve for the equilibrium or resting potential and use this as the initial condition for the following. After threshold depolarization, the action potential may be approximated by three successive stages:

(a) A brief but large Na\(^+\) conductance increase, so let \(g_{Na} = 5.0\) and \(g_K = 0.6\) for \(0 \leq t < 1 \text{ ms}\).

(b) Na\(^+\) conductance decreases to zero and K\(^+\) conductance increases, so \(g_{Na} = 0\) and \(g_K = 2.0\) for \(1 \leq t < 4 \text{ ms}\).

(c) Both conductances return to normal, so \(g_{Na} = 0.1\) and \(g_K = 0.6\) for \(t \leq 4 \text{ ms}\). Calculate the piecewise linear solution to this problem and plot the result. This problem should give you an intuitive feel for the role of Na\(^+\) and K\(^+\) conductance changes in generating the action potential.

5. Generalize eqn (2.19) to the case where there are \(N\) simultaneous excitatory synaptic events. Assuming that there is no inhibition and that \(g_e = 2 \text{ nS}\), the equation becomes:

\[
\frac{dV}{dt} = -\frac{1}{12.5} \{(V + 75) - 2NV\} \quad \text{for } 0 \leq t \leq 1
\]

\(V(0) = -75\)

Obtain the analytical solution for \(0 \leq t \leq 1 \text{ ms}\) (you need not solve for the decay phase). Plot the peak EPSP value \(V(1)\) for \(1 \leq N \leq 12\). Do EPSPs add linearly, or does the biophysics produce saturation effects?

6. Solve eqn (2.18) for the case where there is one excitatory synaptic event but \(N\) concurrent inhibitory events. Assuming that \(g_e = 2 \text{ nS}\), and \(g_i = 12 \text{ nS}\) the equation becomes:

\[
\frac{dV}{dt} = -\frac{1}{12.5} \{((V + 75) + 2V + 12N(V + 75))
\]

\(V(0) = -75\)

Obtain the analytical solution for \(0 \leq t \leq 1 \text{ ms}\) (you need not solve for the decay phase). Plot the peak EPSP value \(V(1)\) for \(0 \leq N \leq 8\). Discuss the results of this shunting inhibition in terms of the linearity or nonlinearity of synaptic effects.